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Optimizing a Two-Warehouse Single Vendor-Buyer Inventory Model with Varying Demand and Varying Holding Costs Under Trade Credit and Imperfect Production

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ABSTRACT

This paper presents a comprehensive inventory model for a single-vendor, single-buyer supply chain functioning within a two-warehouse system, incorporating crucial factors such as trade credit, stock-dependent demand, and variable holding costs. The model addresses the challenges posed by imperfect production processes, enabling an examination of how production defects influence total costs. Demand is considered stock-dependent, reflecting the relationship between inventory levels and market demand. The model includes a trade credit policy, where the vendor offers the buyer a credit period for payment, promoting trust in their business interaction. A more realistic approach is taken by applying stock-dependent holding costs exclusively to the rented warehouse, aligning with typical cost structures observed in practice. The aim of the model is to identify optimal inventory policies that minimize total costs, including setup, holding, transportation, screening, warranty, and purchasing costs. Furthermore, the interest earned during the credit period is deducted from the total cost to account for the effect of the trade credit policy. A detailed cost function is derived, with its convexity demonstrated through graphical analysis to manage the non-linear behavior of the function. Finally, sensitivity analysis is carried out to assess the influence of key parameters on system performance, providing valuable insights into cost reduction opportunities and operational improvements.

INTRODUCTION

In today's rapidly changing market environment, efficient inventory management has become crucial for maintaining a competitive advantage in supply chains. As supply chains become more complex and diverse inventory storage options arise, the need for advanced inventory models that can tackle various operational challenges is growing. One such model that has gained attention is the Joint Economic Lot Size (JELS) method. This approach provides a simplified solution for minimizing costs, making it an appealing choice for businesses looking to improve their inventory management practices. The use of the JELS method is increasing as companies recognize its potential for optimizing operations and reducing overall costs. Groundbreaking researchers like Goyal (1977) and Banerjee (1986) have played a key role in establishing the foundation for this method.

The two-warehouse inventory system has attracted considerable attention as it reflects the practical reality where firms frequently manage both an owned warehouse and a rented warehouse. Owned warehouses typically offer lower holding costs, whereas rented warehouses incur variable charges, making cost optimization more challenging. Additionally, demand in many markets is not constant but influenced by the level of available stock, a phenomenon referred to as stock-dependent demand. Integrating such demand behavior into inventory models provides a closer approximation to consumer purchasing patterns and enables firms to adopt more responsive strategies.

Another critical dimension of inventory management is the role of trade credit policies, where the vendor extends a credit period to the buyer. Trade credit not only fosters stronger buyer–seller relationships but also influences the buyer's effective costs through the interest earned during the credit period. At the same time, the presence of imperfect production processes introduces defects that can increase costs associated with screening, warranty, and replacements. These factors, when considered collectively, present both challenges and opportunities for supply chain cost minimization.

Motivated by these practical considerations, this study develops a comprehensive model for a single-vendor, single-buyer supply chain operating under a two-warehouse system. The model integrates stock-dependent demand, trade credit policy, imperfect production processes, and variable holding costs. In particular, holding costs are assumed to be stock-dependent only for the rented warehouse, reflecting realistic cost structures observed in practice. The model derives a total cost function that incorporates setup, holding, transportation, screening, warranty, and purchasing costs, while also accounting for the financial benefits of trade credit. The convexity of the cost function is demonstrated to ensure the existence of an optimal solution. Furthermore, sensitivity analysis is performed to examine the impact of key parameters, offering managerial insights into cost reduction opportunities and operational improvements.

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LITERATURE REVIEW

In this section, we review the existing literature across four key streams relevant to our study: two-warehouse inventory systems, stock-dependent demand, imperfect production processes, trade credit and variable holding costs.

Two-Warehouse System

Two-warehouse systems typically facilitate cost reduction, enhance supply chain management, and offer secure inventory storage until final delivery to retail outlets. This dual-warehouse strategy is a more practical solution as it enables companies to avoid the high costs associated with maintaining excessive storage space, which can significantly increase operational expenses. Instead, the two-warehouse system provides an efficient alternative for handling surplus inventory. Several researchers have examined two-warehouse inventory systems. For instance, Zhou *et al.* (2005) developed a model incorporating stock-dependent demand, while Chung *et al.* (2009) focused on imperfect production quality under constant demand. Yang and Chang (2012) analyzed a system for deteriorating items with allowable payment delays. Bhunia *et al.* (2014) extended this to deteriorating items with both permissible delays and partial backlogging. Payel *et al.* (2017) integrated stock-dependent demand and imperfect production processes into their model. Panda *et al.* (2018) explored price- and stock-dependent demand alongside trade credit policies under shortage conditions, and Pal *et al.* (2024) examined trade credit policies in the context of inflation within two-warehouse systems. Chowdhury *et al.* (2023) examined warehousing policies in the context of inventory management.

Stock Dependent Demand

Product demand is often influenced by stock levels, which can significantly impact inventory management strategies. Changes in inventory can signal scarcity or surplus, affecting consumer behavior and demand trends. As a result, businesses must carefully monitor stock levels to ensure effective inventory management. Sajadeih *et al.* (2010) proposed a stock-dependent demand model that considers the buyer's display area for inventory placement. Yang *et al.* (2012) studied a two-warehouse system for deteriorating items with permissible payment delays. Similarly, Zhou *et al.* (2005) and Chung *et al.* (2009) investigated stock-dependent demand in their respective models. Khan *et al.* (2022) explored non-linear stock-dependent demand, adding further insight into the field.

Imperfect Quality Production

Imperfect quality production is a common challenge in real-world manufacturing processes, as variability often results in defects. Managing imperfect items is crucial for inventory and quality control, as it directly affects production efficiency, costs, and customer satisfaction. Huang (2002) introduced the first integrated vendor-buyer inventory model that accounted for imperfect quality items. Hsu *et al.* (2012) extended this by incorporating an imperfect screening process at the buyer's end, accounting for type I and type II errors. Wangsa *et al.* (2019) further expanded the analysis by considering inspection errors, freight costs based on shipping weight and distance, and stochastic lead times influenced by both production and transportation durations. Barman (2022) and Kumar (2017) also considered imperfect production in their single-vendor, single-buyer inventory models.

Trade Credit

In practice, buyers often delay payments to vendors, a common business practice encouraged by vendors through extended payment terms. This strategy fosters a more collaborative business environment, strengthening vendor-buyer relationships and improving cash flow management for both parties. Allowing buyers to delay payment helps them invest in growth, manage their expenses more efficiently, and adjust to demand fluctuations. Goyal (1985) pioneered the first Economic Order Quantity (EOQ) model with permissible payment delays, while Chang *et al.* (2008) developed an integrated vendor-buyer model linking trade credit periods to order quantities, showing that extended payment terms can boost supply chain profitability. Sarkar *et al.* (2014) expanded this by incorporating payment delays, defective production, and variable lead times. Pervin *et al.* (2018) introduced a two-level trade credit policy for deteriorating items, considering time-dependent holding costs and stock-dependent demand. Recently, Momena *et al.* (2023) proposed an integrated model accounting for two storage facilities, time-dependent holding costs, and demand influenced by both price and advertising frequency.

Variable Holding Cost

Traditional inventory models often assume constant holding costs over time, but this assumption overlooks the reality that holding costs may vary with time and stock levels. Goh (1992) introduced a stock-dependent demand model with variable holding costs, defining the unit holding cost as a nonlinear continuous function of time. Other models, such as Giri *et al.* (1996), incorporate time-dependent variations in both holding costs and demand functions. Palanivel *et al.* (2013) also explored time-dependent holding costs alongside variable production costs. Yang (2014) investigated an EOQ model with stock-dependent demand and stock-dependent holding costs. Ogunleye *et al.* (2022) developed a nonpre-emptive integer nonlinear goal programming model for solving multi-item inventory problems.

MATERIALS & METHODS

Theoretical Framework

This paper presents an integrated inventory model for a single-vendor, single-buyer system, considering imperfect production processes and stock-dependent demand. A trade credit policy is also included in the framework. The model incorporates a two-warehouse system comprising an owned warehouse and a rented warehouse where the holding cost for the rented warehouse is a function of the stock level stored. Due to the model's non-linearity, convexity could not be proven mathematically, so it was demonstrated graphically using Mathematica 14.0 software.

Notations

Table 1:

Notations	Units	Description
S	$	Vendor's setup cost
A	$/ shipment	Buyer's setup cost
h_v	$/ unit	Holding cost of vendor
h_w	$/ unit	Holding cost of buyer's own warehouse
h_r	$/ unit	Holding cost of buyer's rented warehouse
F	$/ shipment	Transportation cost
k	$/ unit	Unit screening cost
c	$/ unit	Unit purchasing cost of Buyer
s	$/ unit	Unit selling cost of Buyer
v	$/ unit	Warranty cost of each defective items
w	units	Storage capacity of own storage of Buyer
P	Units/ unit time	Vendor's Production rate
x	Units/ unit time	Unit screening rate
ϱ	Constant	fraction of defective items in each shipment
t_1	Unit of time	Time at which the stock in RW reaches to zero
t_2	Unit of time	Time when screening process is completed (for both warehouses)
T	Unit of time	Time length of each shipment
M	Unit of time	Fixed credit period given by vendor
I_e	%	Interest earned by the Buyer
$I_r(t)$	Units	Inventory level of rented warehouse at time t
$I_w(t)$	Units	Inventory level of own warehouse at time t
a	Constant	Demand parameter
b	Constant	Demand Parameter
α	Constant	Holding cost parameter
β	Constant	Holding cost parameter
Decision Variables		
Q	Units	Shipment Quantity
n	Unit of time	Number of shipments

Assumptions

1. Here single vendor and single buyer is considered.
2. The inventory planning horizon is infinite.
3. Replenishment is instantaneous.
4. The vendor produces nQ items at a constant production rate P, ensuring that the production rate is always higher than the demand. The vendor then delivers these items to the buyer in batches of Q units.
5. The vendor's production process is imperfect, resulting in a fraction ϱ $(0\leq\varrho\leq1)$ of defective items in each lot.
6. Buyer's Demand rate is function of inventory levels, as $D(I(t))=a+bI(t)$; where $I(t)=I_r(t)+I_w(t)$ and a and b are positive constants, $a>b$.
7. The buyer utilizes two warehouses: one is an owned warehouse (OW), where the holding cost is fixed h_w , and the other is a rented warehouse (RW), where the holding cost varies based on the inventory levels stored in this facility.
8. The holding cost of rented warehouse is function of the inventory levels of rented warehouse and is assumed as $h_r(I_r(t))=\alpha+\beta I_r(t)$, where α and β are positive constants.) When inventory levels are zero, the holding cost is also zero.
9. The vendor offers a fixed credit period M to the buyer. The buyer is required to settle the account with the vendor upon the completion of this credit period.
10. During the trade credit period , the buyer utilizes the funds generated from selling items to invest, thereby earning an interest I_e on that amount.

11. Upon receiving the order, the stock undergoes a perfect screening process at a rate of x. This screening process is initiated at the rented warehouse, followed by the owned warehouse. *($x >> D(I(t))$).*

12. No shortage is allowed.

13. During the inspection process, the buyer identifies and segregates defective items, which are subsequently returned to the vendor. The vendor compensates the buyer for the warranty cost associated with each defective item.

The Mathematical Model

In this model, the vendor begins by producing nQ items, with each shipment consisting of Q items (where $Q>w$) delivered to the buyer. The buyer stores the items across two warehouses: the owned warehouse *(OW)* and the rented warehouse *(RW)*. The buyer prioritizes selling the items stored in the rented warehouse *(RW)* to minimize holding costs, which are higher in the rented warehouse compared to the owned warehouse *(OW)*. As soon as the inventory is received, the stock in the rented warehouse is depleted first through both sales (due to demand) and inspection. This strategy allows the buyer to reduce the more expensive holding costs associated with the rented warehouse while maintaining inventory in the owned warehouse with a lower fixed holding cost. Following receipt, the buyer initiates a screening process, starting with the rented warehouse and subsequently the owned warehouse, with the screening process taking a total time of t_2. The inventory in the rented warehouse reaches zero after t_1 time. Two distinct cases arise based on the relationship between t_2 and t_1: one where $t_2 \leq t_1$, and another where $t_1 \leq t_2$.

Case 1. $t_2 \leq t_1$

In the case where t_2≤t_1, it implies that the inventory in the rented warehouse takes an equal or longer time to be fully depleted compared to the time required for the screening process. Once the stock reaches the buyer, both the inspection and stock depletion due to demand begin simultaneously.

We first focus on the inventory level of the rented warehouse *(RW)*. From the start until time t_2, the change in inventory is depended on both time and the screening process. Therefore, the rate of change in the inventory level of the rented warehouse is:

$$(dI_r(t))/dt=-a-b(I_r(t)+I_w(t))-x\varrho, \quad 0\leq t\leq t_2 \qquad (1)$$

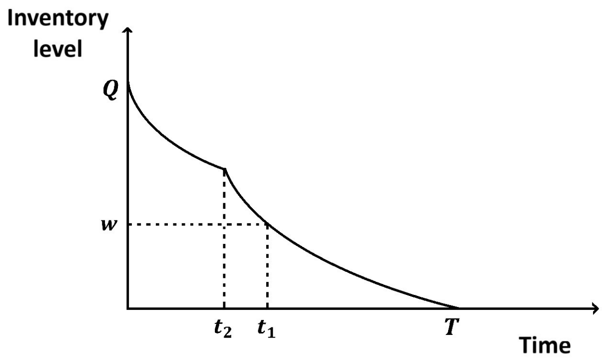

Figure 1: Inventory level of buyer for first case

After time t_2, the defective items have been separated and removed, and the inventory in the rented warehouse depends solely on the demand.

$$(dI_r(t))/dt=-a-b(I_r(t)+I_w(t)), \quad t_2\leq t\leq t_1 \qquad (2)$$

Above differential equation subjected to conditions: I_r $(0)=Q-w$, $I_r(t_1)=0$

On solving the Equations (1) and (2) and using this conditions, one can get

$$I_r(t)=(Q-w)e^{-bt}-((a+bw+x\varrho))/b\,(1-e^{-bt}), \quad 0\leq t\leq t_2$$

$$I_r(t)=(a+bw)/b\,(e^{b(t1-t)}-1), \quad t_2\leq t\leq t$$

Again, using equation (1) and (2), continuity of I_r (t) at time t_2, one can get

$$t_1=1/b\ \log(e^{bt2)}+(b(Q-w)/(a+bw))-((a+bw+x\varrho))/(a+bw)\,(e^{-bt2}-1))$$

For the owned warehouse (OW), the inventory level remains unchanged from the start until the completion of the screening process in the rented warehouse.

$$I_w=w, \quad 0\leq t\leq t_2-w/x \qquad (3)$$

After the completion of the screening process in the rented warehouse, the inventory levels are adjusted to account for the separation of defective items (screening process).

$$I_w=w-\varrho wt, \quad t_2-w/x\leq t\leq t_2 \qquad (4)$$

From time t_2 to time t_1, the inventory level in the system remains w-ρw

$$I_w=w-\varrho w, \quad t_2\leq t\leq t_1 \qquad (5)$$

From t_1 to T, the inventory level solely depends upon the demand

$$dI_w(t)/dt=-a-bI_w(t), \quad t_1\leq t\leq T \qquad (6)$$

Subject to conditions, $I_w\, t_1=w-\varrho w$, $I_w(T)=0$

Solving equation (6), using condition, one can get

$$I_w=a/b\,(e^{b(T-M)}-1), \quad t_1\leq t\leq T$$

$$T=1/b\ \log(a+bw(1-\varrho)/a)+t_1$$

Costs of the Buyer

Holding cost of rented warehouse $=\int_0^{t1} h_r.I_r(t)dt$

$$= \int_0^{t_1} (\alpha + \beta I_r) \cdot I_r(t)dt$$

$$|= \int_0^{t_2} (\alpha + \beta I_r) \cdot I_r(t)dt + \int_{t_2}^{t_1} (\alpha + \beta I_r) \cdot I_r(t)dt$$

$$= \alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_2}) - \frac{(a+bw+x\rho)}{b}\left(t_2 + \frac{e^{-bt_2}-1}{b}\right) + \frac{(a+bw)}{b}\left(\frac{1-e^{b(t_1-t_2)}}{b} - t_1 + t_2\right)\right)$$
$$+ \beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_2}) + \frac{(a+bw+x\rho)^2}{b^2}\left(t_2 + \frac{1-e^{-2bt_2}}{2b} + \frac{2(e^{-bt_2}-1)}{b}\right)\right.$$
$$- \frac{(Q-w)(a+bw+x\rho)}{b^2}\left(1 - e^{-bt_2} - \frac{e^{-2bt_2}-1}{2}\right)$$
$$\left. + \frac{(a+bw)^2}{b^2}\left(t_1 - t_2 + \frac{1-e^{2b(t_1-t_2)}}{2b} - \frac{2(1-e^{b(t_1-t_2)})}{b}\right)\right)$$

Then, the holding cost of owned warehouse $= h_w \int_0^T I_w\,(t)$

$$= h_w \cdot \int_0^{t_2 - \frac{w}{x}} I_w(t) + h_w \cdot \int_{t_2 - \frac{w}{x}}^{t_2} I_w(t) + h_w \cdot \int_{t_2}^{t_1} I_w(t) + h_w \cdot \int_{t_1}^{T} I_w(t)$$

$$= h_w \cdot ((w - yw)t_1 + \rho w t_2) + h_w \cdot \rho w\left(\frac{w^2}{x} - \frac{wt_2}{2x^2}\right) + h_w \cdot \frac{a}{b}\left(\frac{e^{b(T-t_1)}-1}{b} - T + t_1\right)$$

The vendor offers a trade credit period, denoted as M, to the buyer. Consequently, this leads to the emergence of five following distinct sub-cases

Sub-case 1. *T≤M*

Then interest earned during time period $M = sI_e\left(\int_0^T (D(I(t)).tdt + (M-T)Q\right)$

$$= sI_e\left(-(Q-w)\left(t_2 e^{-bt_2} + \frac{1-e^{-bt_2}}{b}\right) - \frac{x\rho t_2^2}{2} + \frac{b\rho w^3 t_2}{x^2} - \frac{b\rho w^4}{3x^3} - \frac{b\rho w^2 t_2^2}{x} + \frac{bwt_1^2}{2} - b\rho w\left(\frac{t_1^2}{2} - \frac{t_2^2}{2}\right)\right.$$
$$+ (a+bw+x\rho)\left(-\frac{t_2 e^{-bt_2}}{b} + \frac{(1-e^{-bt_2})}{b^2}\right)(a+bw)\left(-\frac{t_1}{b} + \frac{t_2 e^{b(t_1-t_2)}}{b}\right.$$
$$\left.\left. - \frac{(1-e^{b(t_1-t_2)})}{b^2}\right) - \frac{t_1^2 bw}{2} + a\left(\frac{t_1 e^{(T-t_1)}}{b} - \frac{T}{b} + \frac{(e^{b(T-t_1)}-1)}{b^2}\right) + (M-T)Q\right)$$

Sub-case 3. $t_2 < M \le t_1$

Then interest earned during time period $M = sI_e\left(\int_0^M (D(I(t)).tdt\right)$

$$= sI_e\left(-(Q-w)\left(t_2 e^{-bt_2} + \frac{1-e^{-bt_2}}{b}\right) - x\rho t_2^2 + \frac{b\rho w^3 t_2}{x^2} - \frac{b\rho w^4}{3x^3} - \frac{b\rho w^2 t_2^2}{x} + \frac{bwM^2}{2} - byw\left(\frac{M^2}{2} - \frac{t_2^2}{2}\right)\right.$$
$$+ (a+bw+x\rho)\left(-\frac{t_2 e^{-bt_2}}{b} + \frac{(1-e^{-bt_2})}{b^2}\right)$$
$$\left. + (a+bw)\left(-\frac{Me^{b(t_1-M)}}{b} + \frac{t_2 e^{b(t_1-t_2)}}{b} - \frac{(e^{b(t_1-M)} - e^{b(t_1-t_2)})}{b^2}\right)\right)$$

Sub-case 4. $t_2 - w/x < M \le t_2$

Then interest earned during time period $M = sI_e\left(\int_0^M D(I(t)).tdt\right)$

$$= sI_e\left(-(Q-w)\left(t_2 e^{-bM} + \frac{(e^{-bM}-1)}{b}\right) - \frac{x\rho M^2}{2} - \frac{b\rho wM^3}{3} + \frac{b\rho w t_2^3}{3} - \frac{b\rho w^2 t_2^2}{x} - \frac{b\rho w^4}{3x^3} + \frac{bw^3 t_2}{x^2}\right.$$
$$\left. + (a+bw+x\rho)\left(-\frac{Me^{-bM}}{b} + \frac{(1-e^{bM})}{b^2}\right) - \frac{b\rho wM^3}{3}\right)$$

Sub-case 5. $0 < M \le t_2 - w/x$

Then interest earned during time period $M = sI_e\left(\int_0^M D(I(t)).tdt\right)$

$$= sI_e\left(-(Q-w)\left(t_2 e^{-bM} + \frac{(e^{-bM}-1)}{b}\right) - \frac{x\rho M^2}{2} + (a+bw+x\rho)\left(-\frac{Me^{-bM}}{b} + \frac{(1-e^{bM})}{b^2}\right)\right)$$

Then, the total cost of buyer is given by

$TC_B\,(n,Q)$= Setup cost + Screening cost + Holding cost + Purchasing cost + Transportation cost - Interest earned

Costs of the Vendor

The total cost of vendor is given by

$TC_V\,(n,Q)$= Setup cost + Warranty cost + Holding cost

$TC_V(n,Q)=S+nvQ\rho+h_v((nQ^2)/P-(n^2Q^2)/2P+QTn(n-1)/2)$

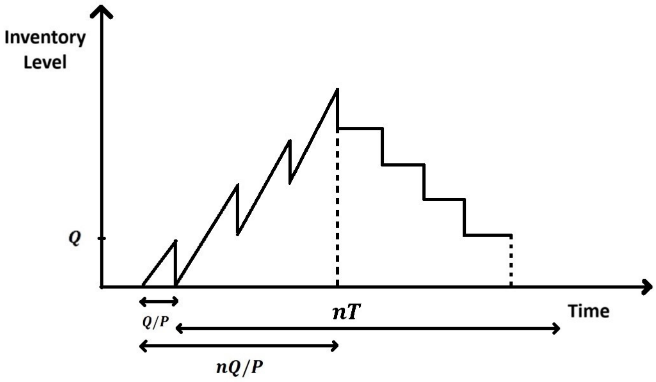

Figure 2: Inventory level of Vendor

Total Cost Per Unit Time of Integrated Systems

The total cost per unit time, which is the sum of the costs incurred by both the vendor and the buyer unit time, can be expressed as follows:

For $T\leq M$

$$
\begin{aligned}
TC_2(n,Q) = \frac{1}{nT}\Bigg(& S + nvQ\rho + h_v\left(\frac{nQ^2}{P} - \frac{n^2Q^2}{2P} + \frac{QTn(n-1)}{2}\right) + ncQ + nA + nF + nkQ \\
& + n\alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_2}) - \frac{(a+bw+x\rho)}{b}\left(t_2 + \frac{e^{-bt_2}-1}{b}\right)\right. \\
& \left. + \frac{(a+bw)}{b}\left(\frac{1-e^{b(t_1-t_2)}}{b} - t_1 + t_2\right)\right) \\
& + n\beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_2}) + \frac{(a+bw+x\rho)^2}{b^2}\left(t_2 + \frac{1-e^{-2bt_2}}{2b} + \frac{2(e^{-bt_2}-1)}{b}\right)\right. \\
& - \frac{(Q-w)(a+bw+x\rho)}{b^2}\left(1 - e^{-bt_2} - \frac{e^{-2bt_2}-1}{2}\right) \\
& \left. + \frac{(a+bw)^2}{b^2}\left(t_1 - t_2 + \frac{1-e^{2b(t_1-t_2)}}{2b} - \frac{2\left(1-e^{b(t_1-t_2)}\right)}{b}\right)\right) + h_w\cdot((w-\rho w)t_1 + \rho wt_2) \\
& + h_w\cdot bw\left(\frac{w^2}{x} - \frac{wt_2}{2x^2}\right) + h_w\cdot\frac{a}{b}\left(\frac{e^{b(T-t_1)}-1}{b} - T + t_1\right) \\
& - nsI_e\left(-(Q-w)\left(t_2e^{-bt_2} + \frac{1-e^{-bt_2}}{b}\right) - \frac{x\rho t_2^2}{2} + \frac{b\rho w^3t_2}{x^2} - \frac{b\rho w^4}{3x^3} - \frac{b\rho w^2t_2^2}{x} + \frac{bwt_1^2}{2}\right. \\
& - b\rho w\left(\frac{t_1^2}{2} - \frac{t_2^2}{2}\right) + (a+bw+x\rho)\left(-\frac{t_2e^{-bt_2}}{b} + \frac{(1-e^{-bt_2})}{b^2}\right) \\
& + (a+bw)\left(-\frac{t_1}{b} + \frac{t_2e^{b(t_1-t_2)}}{b} - \frac{\left(1-e^{b(t_1-t_2)}\right)}{b^2}\right) - \frac{t_1^2bw}{2} \\
& \left. + a\left(-\frac{Me^{b(T-M)}}{b} + \frac{t_1e^{b(T-t_1)}}{b} + \frac{\left(e^{b(T-t_1)} - e^{b(T-M)}\right)}{b^2}\right)\right)\Bigg)
\end{aligned}
$$

For $t_2 < M \le t_1$

$$
\begin{aligned}
TC_3(n,Q) = \frac{1}{nT}\Bigg(& S + nvQ\rho + h_v\left(\frac{nQ^2}{P} - \frac{n^2Q^2}{2P} + \frac{QTn(n-1)}{2}\right) + ncQ + nA + nF + nkQ \\
& + n\alpha\left(\frac{(Q-w)}{b}\left(1-e^{-bt_2}\right) - \frac{(a+bw+x\rho)}{b}\left(t_2 + \frac{e^{-bt_2}-1}{b}\right)\right. \\
& \left. + \frac{(a+bw)}{b}\left(\frac{1-e^{b(t_1-t_2)}}{b} - t_1 + t_2\right)\right) \\
& + n\beta\left(\frac{(Q-w)^2}{2b}\left(1-e^{-2bt_2}\right) + \frac{(a+bw+x\rho)^2}{b^2}\left(t_2 + \frac{1-e^{-2bt_2}}{2b} + \frac{2\left(e^{-bt_2}-1\right)}{b}\right)\right. \\
& - \frac{(Q-w)(a+bw+x\rho)}{b^2}\left(1 - e^{-bt_2} - \frac{e^{-2bt_2}-1}{2}\right) \\
& \left. + \frac{(a+bw)^2}{b^2}\left(t_1 - t_2 + \frac{1-e^{2b(t_1-t_2)}}{2b} - \frac{2\left(1-e^{b(t_1-t_2)}\right)}{b}\right)\right) + h_w\cdot\left((w-\rho w)t_1 + ywt_2\right) \\
& + h_w\cdot bw\left(\frac{w^2}{x} - \frac{wt_2}{2x^2}\right) + h_w\cdot\frac{a}{b}\left(\frac{e^{b(T-t_1)}-1}{b} - T + t_1\right) \\
& - nsI_e\left(-(Q-w)\left(t_2e^{-bt_2} + \frac{1-e^{-bt_2}}{b}\right) - x\rho t_2^2 + \frac{b\rho w^3t_2}{x^2} - \frac{b\rho w^4}{3x^3} - \frac{b\rho w^2t_2^2}{x} + \frac{bwM^2}{2}\right. \\
& - byw\left(\frac{M^2}{2} - \frac{t_2^2}{2}\right) + (a+bw+x\rho)\left(-\frac{t_2e^{-bt_2}}{b} + \frac{\left(1-e^{-bt_2}\right)}{b^2}\right) \\
& \left.\left. + (a+bw)\left(-\frac{Me^{b(t_1-M)}}{b} + \frac{t_2e^{b(t_1-t_2)}}{b} - \frac{\left(e^{b(t_1-M)} - e^{b(t_1-t_2)}\right)}{b^2}\right)\right)\right)
\end{aligned}
$$

For $t_2 - w/x < M \le t_2$

$$
\begin{aligned}
TC_4(n,Q) = \frac{1}{nT}\Bigg(& S + nvQ\rho + h_v\left(\frac{nQ^2}{P} - \frac{n^2Q^2}{2P} + \frac{QTn(n-1)}{2}\right) + ncQ + nA + nF + nkQ \\
& + n\alpha\left(\frac{(Q-w)}{b}\left(1-e^{-bt_2}\right) - \frac{(a+bw+x\rho)}{b}\left(t_2 + \frac{e^{-bt_2}-1}{b}\right)\right. \\
& \left. + \frac{(a+bw)}{b}\left(\frac{1-e^{b(t_1-t_2)}}{b} - t_1 + t_2\right)\right) \\
& + n\beta\left(\frac{(Q-w)^2}{2b}\left(1-e^{-2bt_2}\right) + \frac{(a+bw+x\rho)^2}{b^2}\left(t_2 + \frac{1-e^{-2bt_2}}{2b} + \frac{2\left(e^{-bt_2}-1\right)}{b}\right)\right. \\
& - \frac{(Q-w)(a+bw+x\rho)}{b^2}\left(1 - e^{-bt_2} - \frac{e^{-2bt_2}-1}{2}\right) \\
& \left. + \frac{(a+bw)^2}{b^2}\left(t_1 - t_2 + \frac{1-e^{2b(t_1-t_2)}}{2b} - \frac{2\left(1-e^{b(t_1-t_2)}\right)}{b}\right)\right) + h_w\cdot\left((w-yw)t_1 + ywt_2\right) \\
& + h_w\cdot bw\left(\frac{w^2}{x} - \frac{wt_2}{2x^2}\right) + h_w\cdot\frac{a}{b}\left(\frac{e^{b(T-t_1)}-1}{b} - T + t_1\right) \\
& - nsI_e\left(\frac{aM^2}{2} - (Q-w)\left(t_2e^{-bM} + \frac{\left(e^{-bM}-1\right)}{b}\right) - -\frac{x\rho M^2}{2} - \frac{b\rho wM^3}{3} + \frac{b\rho wt_2^3}{3} - \frac{b\rho w^2t_2^2}{x}\right. \\
& \left.\left. - \frac{b\rho w^4}{3x^3} + \frac{bw^3t_2}{x^2} + (a+bw+x\rho)\left(-\frac{Me^{-bM}}{b} + \frac{\left(1-e^{bM}\right)}{b^2}\right) - \frac{bwM^3}{3}\right)\right)
\end{aligned}
$$

For $0<M\leq t_2-w/x$

$$
\begin{aligned}
TC_5(n,Q) = \frac{1}{nT}\Bigg(& S + nvQ\rho + h_v\left(\frac{nQ^2}{P} - \frac{n^2Q^2}{2P} + \frac{QTn(n-1)}{2}\right) + ncQ + nA + nF + nkQ \\
& + n\alpha\Bigg(\frac{(Q-w)}{b}(1-e^{-bt_2}) - \frac{(a+bw+x\rho)}{b}\left(t_2 + \frac{e^{-bt_2}-1}{b}\right) \\
& + \frac{(a+bw)}{b}\left(\frac{1-e^{b(t_1-t_2)}}{b} - t_1 + t_2\right)\Bigg) \\
& + n\beta\Bigg(\frac{(Q-w)^2}{2b}(1-e^{-2bt_2}) + \frac{(a+bw+x\rho)^2}{b^2}\left(t_2 + \frac{1-e^{-2bt_2}}{2b} + \frac{2(e^{-bt_2}-1)}{b}\right) \\
& - \frac{(Q-w)(a+bw+x\rho)}{b^2}\left(1-e^{-bt_2} - \frac{e^{-2bt_2}-1}{2}\right) \\
& + \frac{(a+bw)^2}{b^2}\left(t_1 - t_2 + \frac{1-e^{2b(t_1-t_2)}}{2b} - \frac{2\left(1-e^{b(t_1-t_2)}\right)}{b}\right)\Bigg) + h_w\cdot\left((w-\rho w)t_1 + ywt_2\right) \\
& + h_w\cdot bw\left(\frac{w^2}{x} - \frac{wt_2}{2x^2}\right) + h_w\cdot\frac{a}{b}\left(\frac{e^{b(T-t_1)}-1}{b} - T + t_1\right) \\
& - nsI_e\Bigg(-(Q-w)\left(t_2e^{-bM} + \frac{(e^{-bM}-1)}{b}\right) - \frac{x\rho M^2}{2} \\
& + (a+bw+x\rho)\left(-\frac{Me^{-bM}}{b} + \frac{(1-e^{bM})}{b^2}\right)\Bigg)\Bigg)
\end{aligned}
$$

Case 2. $t_1\leq t_2$

In this case, where, the screening process in the regular warehouse is completed after the stock in the rented warehouse has been fully depleted. This situation can also be described as the rented warehouse holding fewer items compared to the owned warehouse.

So, the inventory level of Rented warehouse follows the differential equation as

$dI_r(t)/dt=-a-b(I_r(t)+I_w(t))-x\rho,\qquad 0\leq t\leq t_1\qquad (7)$

Subject to the conditions, $I_r(0)=Q-w,\ I_r(t_1)=0$

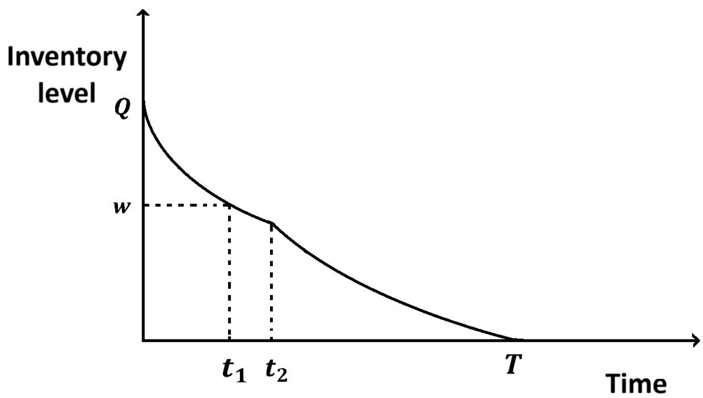

Figure 3: Inventory level of buyer for second case

Solving equation (7), using conditions, one can get

$$I_r(t) = (Q-w)e^{-bt} - \frac{(a+bw+x\rho)}{b}(1-e^{-bt})$$

$$t_1 = \frac{1}{b}\log\left(\frac{Q-w}{(a+bw+x\rho)b} + 1\right)$$

As the inventory in the rented warehouse decreases, the time required to inspect the remaining items will correspondingly reduce. To simplify the analysis, we assume that the screening process in the rented warehouse is halted once the inspection is complete, and it only resumes when the stock in the rented warehouse is entirely depleted. Here t_2 can be written as t_1+w/x.

Similarly for owned warehouse inventory level Follows differential equation as

$$\frac{dI_w(t)}{dt} = -a - bI_w(t) - x\rho,\qquad t_1\leq t\leq t_2 \qquad (8)$$

$$\frac{dI_w(t)}{dt} = -a - bI_w(t),\qquad t_2\leq t\leq T \qquad (9)$$

Subject to conditions: $I_w(t_1)=w,\ I_w(T)=0$

Solving the equation (8) and (9), using conditions, one can get

$$I_w(t) = we^{b(t_1-t)} - \frac{(a+x\rho)}{b}\cdot\left(1-e^{b(t_1-t)}\right)$$

$$I_w(t) = \frac{a}{b}\left(e^{b(T-t)}-1\right)$$

From the continuity of $I_w(t)$ at t_2,

$$T = \frac{1}{b}\log\left(\frac{e^{bt_1}(bw + a + x\rho) - x\rho e^{bt_2}}{a}\right)$$

Costs of the Buyer

The holding cost of rented warehouse $=\int_0^{t_1} h_r \cdot I_r(t)dt$

$$= \alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_1}) - (a+bw+x\rho)\left(t_1 + \frac{e^{-bt_1}}{b} - \frac{1}{b}\right)\right) + \beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_1}) + \frac{(a+bw+x\rho)^2}{b^2}\left(t_1 - \frac{e^{-2bt_1}}{2b} + \frac{2e^{-bt_1}}{b} - \frac{3}{2b}\right) - \frac{2(Q-w)(a+bw+x\rho)}{b^2}\left(\frac{e^{-2bt_1}}{2} - e^{-bt_1} + \frac{1}{2}\right)\right)$$

And holding cost of owned warehouse $= h_w \cdot \int_0^T I_w(t)dt$

$$= h_w \cdot \left(-\frac{we^{b(t_1-t_2)}}{b} - \frac{(a+x\rho)}{b}\left(t_2 + e^{b(t_1-t_2)}\right) + \frac{w}{b} + \frac{(a+x\rho)}{b}\left(t_1 + \frac{1}{b}\right) + \frac{a}{b}\left(-\frac{1}{b} - T + \frac{e^{b(T-t_2)}}{b} + t_2\right)\right)$$

Similar to Case 1, Also four following sub-cases arise-

Sub-case 1. $T \le M$

Then interest earned during time period M

$$= sI_e\left(\int_0^T D(I(t)) \cdot tdt + (M-T)Q\right)$$

$$= sI_e\left((Q-w)\left(-t_1e^{-bt_1} - \frac{1}{b}(e^{-bt_1}-1)\right) - (a+bw+x\rho)\left(\frac{e^{-bt_1}}{b^2} + \frac{t_1e^{-bt_1}}{b} - t_2 \cdot \frac{e^{b(t_1-t_2)}}{b} + \frac{t_1}{b} - \frac{e^{b(t_1-t_2)}}{b^2}\right) - \frac{x\rho t_2^2}{2} + a\left(\frac{t_2e^{b(T-t_2)}}{b} - \frac{T}{b} - \frac{1}{b^2} + \frac{e^{b(T-t_2)}}{b^2}\right) + (M-T)Q\right)$$

Sub-case 2. $t_2 < M \le T$

Then interest earned during time period M

$$= sI_e\left((Q-w)\left(-t_1e^{-bt_1} - \frac{1}{b}(e^{-bt_1}-1)\right) - (a+bw+x\rho)\left(\frac{e^{-bt_1}}{b^2} + \frac{t_1e^{-bt_1}}{b} - t_2 \cdot \frac{e^{b(t_1-t_2)}}{b} + \frac{t_1}{b} - \frac{e^{b(t_1-t_2)}}{b^2}\right) - \frac{x\rho t_2^2}{2} + a\left(-M \cdot \frac{e^{b(T-M)}}{b} + \frac{t_2e^{b(T-t_2)}}{b} - \frac{e^{b(T-M)}}{b^2} + \frac{e^{b(T-t_2)}}{b}\right)\right)$$

Sub-case 3. $t_1 < M \le t_2$

Then interest earned during time period M

$$= sI_e\left((Q-w)\left(-t_1e^{-bt_1} - \frac{1}{b}(e^{-bt_1}-1)\right) - (a+bw+x\rho)\left(\frac{e^{-bt_1}}{b^2} + \frac{t_1e^{-bt_1}}{b} - \frac{Me^{b(t_1-M)}}{b} + \frac{t_1}{b} - \frac{e^{b(t_1-M)}}{b^2}\right) - \frac{x\rho M^2}{2}\right)$$

Sub-case 4. $0 < M \le t_1$

$$= sI_e\left((Q-w)\left(-Me^{-bM} - \frac{1}{b}(e^{-bM}-1)\right) - (a+bw+x\rho)\left(\frac{e^{-bM}}{b^2} + \frac{Me^{-bM}}{b} - \frac{1}{b^2}\right)\right)$$

Costs of the Vendor

Similar to case 1, total cost of vendor is given by

TC_V= Setup cost + Warranty cost + Holding cost

$$TC_V = S + nvQ\rho + h_v\left(\frac{nQ^2}{P} - \frac{n^2Q^2}{2P} + \frac{QTn(n-1)}{2}\right)$$

and total cost of buyer given by

$TC_B(n,Q)$= Setup cost + Screening cost + Holding cost +

Purchasing cost + Transportation cost - Interest earned unit time is become as

Total Cost Per Unit Time of Integrated Systems For $T \leq M$

So, the total cost (sum of costs of vendor and buyer) per

$$TC_6(n,Q)=\frac{1}{nT}\left(S+nvQ\rho+h_v\left(\frac{nQ^2}{P}-\frac{n^2Q^2}{2P}+\frac{QTn(n-1)}{2}\right)+ncQ+nA+nF+nkQ\right.$$
$$+n\alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_1})-(a+bw+x\rho)\left(t_1+\frac{e^{-bt_1}}{b}-\frac{1}{b}\right)\right)$$
$$+n\beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_1})+\frac{(a+bw+x\rho)^2}{b^2}\left(t_1-\frac{e^{-2bt_1}}{2b}+\frac{2e^{-bt_1}}{b}-\frac{3}{2b}\right)\right.$$
$$\left.-\frac{2(Q-w)(a+bw+x\rho)}{b^2}\left(\frac{e^{-2bt_1}}{2}-e^{-bt_1}+\frac{1}{2}\right)\right)+h_w$$
$$\cdot\left(-\frac{we^{b(t_1-t_2)}}{b}-\frac{(a+x\rho)}{b}\left(t_2+e^{b(t_1-t_2)}\right)+\frac{w}{b}+\frac{(a+x\rho)}{b}\left(t_1+\frac{1}{b}\right)\right.$$
$$\left.+\frac{a}{b}\left(-\frac{1}{b}-T+\frac{e^{b(T-t_2)}}{b}+t_2\right)\right)$$
$$-nsI_e\left((Q-w)\left(-t_1e^{-bt_1}-\frac{1}{b}(e^{-bt_1}-1)\right)\right.$$
$$-(a+bw+x\rho)\left(\frac{e^{-bt_1}}{b^2}+\frac{t_1e^{-bt_1}}{b}-t_2\cdot\frac{e^{b(t_1-t_2)}}{b}+\frac{t_1}{b}-\frac{e^{b(t_1-t_2)}}{b^2}\right)-\frac{x\rho t_2^2}{2}$$
$$\left.\left.+a\left(\frac{t_2e^{b(T-t_2)}}{b}-\frac{T}{b}-\frac{1}{b^2}+\frac{e^{b(T-t_2)}}{b^2}\right)+(M-T)Q\right)\right)$$

For $t_2 < M \leq T$

$$TC_7(n,Q)=\frac{1}{nT}\left(S+nvQ\rho+h_v\left(\frac{nQ^2}{P}-\frac{n^2Q^2}{2P}+\frac{QTn(n-1)}{2}\right)+ncQ+nA+nF+nkQ\right.$$
$$+n\alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_1})-(a+bw+x\rho)\left(t_1+\frac{e^{-bt_1}}{b}-\frac{1}{b}\right)\right)$$
$$+n\beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_1})+\frac{(a+bw+x\rho)^2}{b^2}\left(t_1-\frac{e^{-2bt_1}}{2b}+\frac{2e^{-bt_1}}{b}-\frac{3}{2b}\right)\right.$$
$$\left.-\frac{2(Q-w)(a+bw+x\rho)}{b^2}\left(\frac{e^{-2bt_1}}{2}-e^{-bt_1}+\frac{1}{2}\right)\right)+h_w$$
$$\cdot\left(-\frac{we^{b(t_1-t_2)}}{b}-\frac{(a+x\rho)}{b}\left(t_2+e^{b(t_1-t_2)}\right)+\frac{w}{b}+\frac{(a+x\rho)}{b}\left(t_1+\frac{1}{b}\right)\right.$$
$$\left.+\frac{a}{b}\left(-\frac{1}{b}-T+\frac{e^{b(T-t_2)}}{b}+t_2\right)\right)$$
$$-nsI_e\left((Q-w)\left(-t_1e^{-bt_1}-\frac{1}{b}(e^{-bt_1}-1)\right)\right.$$
$$-(a+bw+x\rho)\left(\frac{e^{-bt_1}}{b^2}+\frac{t_1e^{-bt_1}}{b}-t_2\cdot\frac{e^{b(t_1-t_2)}}{b}+\frac{t_1}{b}-\frac{e^{b(t_1-t_2)}}{b^2}\right)-\frac{x\rho t_2^2}{2}$$
$$\left.\left.+a\left(-M\cdot\frac{e^{b(T-M)}}{b}+\frac{t_2e^{b(T-t_2)}}{b}-\frac{e^{b(T-M)}}{b^2}+\frac{e^{b(T-t_2)}}{b}\right)\right)\right)$$

For $t_1<M\leq t_2$

$$TC_8(n,Q)=\frac{1}{nT}\Bigg(S+nvQ\rho+h_v\left(\frac{nQ^2}{P}-\frac{n^2Q^2}{2P}+\frac{QTn(n-1)}{2}\right)+ncQ+nA+nF+nkQ$$
$$+n\alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_1})-(a+bw+x\rho)\left(t_1+\frac{e^{-bt_1}}{b}-\frac{1}{b}\right)\right)$$
$$+n\beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_1})+\frac{(a+bw+x\rho)^2}{b^2}\left(t_1-\frac{e^{-2bt_1}}{2b}+\frac{2e^{-bt_1}}{b}-\frac{3}{2b}\right)\right.$$
$$\left.-\frac{2(Q-w)(a+bw+x\rho)}{b^2}\left(\frac{e^{-2bt_1}}{2}-e^{-bt_1}+\frac{1}{2}\right)\right)+h_w$$
$$\cdot\left(-\frac{we^{b(t_1-t_2)}}{b}-\frac{(a+x\rho)}{b}\left(t_2+e^{b(t_1-t_2)}\right)+\frac{w}{b}+\frac{(a+x\rho)}{b}\left(t_1+\frac{1}{b}\right)\right.$$
$$\left.+\frac{a}{b}\left(-\frac{1}{b}-T+\frac{e^{b(T-t_2)}}{b}+t_2\right)\right)$$
$$-nsI_e\left((Q-w)\left(-t_1e^{-bt_1}-\frac{1}{b}(e^{-bt_1}-1)\right)\right.$$
$$\left.-(a+bw+x\rho)\left(\frac{e^{-bt_1}}{b^2}+\frac{t_1e^{-bt_1}}{b}-\frac{Me^{b(t_1-M)}}{b}+\frac{t_1}{b}-\frac{e^{b(t_1-M)}}{b^2}\right)-\frac{xyM^2}{2}\right)\Bigg)$$

For $0<M\leq t_1$

$$TC_9(n,Q)=\frac{1}{nT}\Bigg(S+nvQ\rho+h_v\left(\frac{nQ^2}{P}-\frac{n^2Q^2}{2P}+\frac{QTn(n-1)}{2}\right)+ncQ+nA+nF+nkQ$$
$$+n\alpha\left(\frac{(Q-w)}{b}(1-e^{-bt_1})-(a+bw+x\rho)\left(t_1+\frac{e^{-bt_1}}{b}-\frac{1}{b}\right)\right)$$
$$+n\beta\left(\frac{(Q-w)^2}{2b}(1-e^{-2bt_1})+\frac{(a+bw+x\rho)^2}{b^2}\left(t_1-\frac{e^{-2bt_1}}{2b}+\frac{2e^{-bt_1}}{b}-\frac{3}{2b}\right)\right.$$
$$\left.-\frac{2(Q-w)(a+bw+x\rho)}{b^2}\cdot\left(\frac{e^{-2bt_1}}{2}-e^{-bt_1}+\frac{1}{2}\right)\right)+h_w$$
$$\cdot\left(-\frac{we^{b(t_1-t_2)}}{b}-\frac{(a+x\rho)}{b}\left(t_2+e^{b(t_1-t_2)}\right)+\frac{w}{b}+\frac{(a+x\rho)}{b}\left(t_1+\frac{1}{b}\right)+\frac{a}{b}\right.$$
$$\left.\cdot\left(-\frac{1}{b}-T+\frac{e^{b(T-t_2)}}{b}+t_2\right)\right)$$
$$-nsI_e\left((Q-w)\left(-Me^{-bM}-\frac{1}{b}(e^{-bM}-1)\right)-(a+bw+x\rho)\left(\frac{e^{-bM}}{b^2}+\frac{Me^{-bM}}{b}-\frac{1}{b^2}\right)\right)\Bigg)$$

Solution Procedure

The highly nonlinear nature of the equations makes it difficult to mathematically prove the optimality of the solution. To overcome this challenge, we utilized Mathematica 14.0 software to solve the problem and produce a 3D graphical plot. This visualization effectively illustrates the convexity of the solution space, facilitating a clearer understanding of optimal solutions within the framework of the model.

RESULTS & DISCUSSION

Numerical Example

1. S=500000, A=90000, h_v=2, h_w=4, F=1000, k=0.5, c=100, s=120, v=100, w=800, P=5000, x=100000, **ρ**=0.05, M=1, I_e=0.01, a=300, b=0.4, α=5.5, β=0.001.

By using Mathematica, we obtain the minimum cost TC* *(n,Q)*=114006.16, where n*=12 and Q*=1448.

The graphs show the convexity of total cost per unit time function.

2. S=48000, A=6000, h_v=2, h_w=4, F=120, k=0.4, c=30, s=36, v=35, w=300, P=880, x=28000, ρ=0.04, M=2, I_e=0.06, a=70, b=0.4, α=4.5, β=0.001.

By using Mathematica, we obtain the minimum cost TC* *(n,Q)*=10388.92, where n*=8 and *Q*=328*.

The graphs show the convexity of total cost per unit time function.

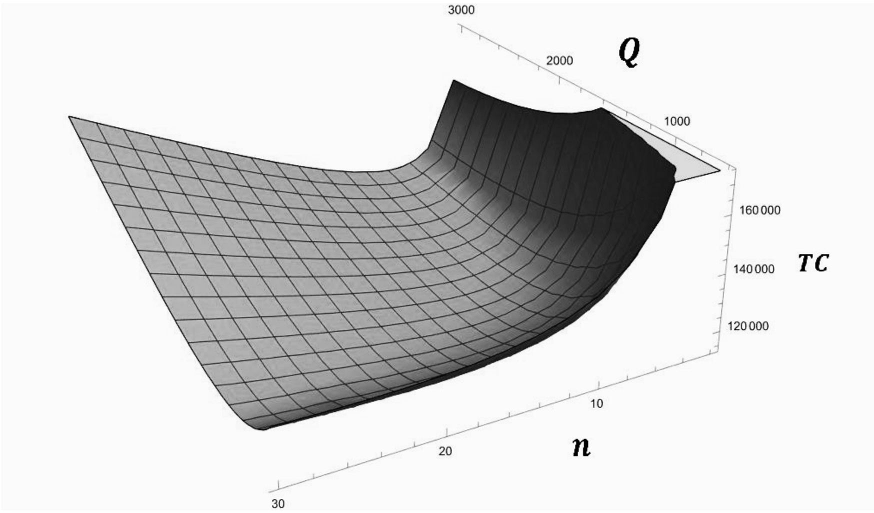

Figure 4: 3D graph of Example 1

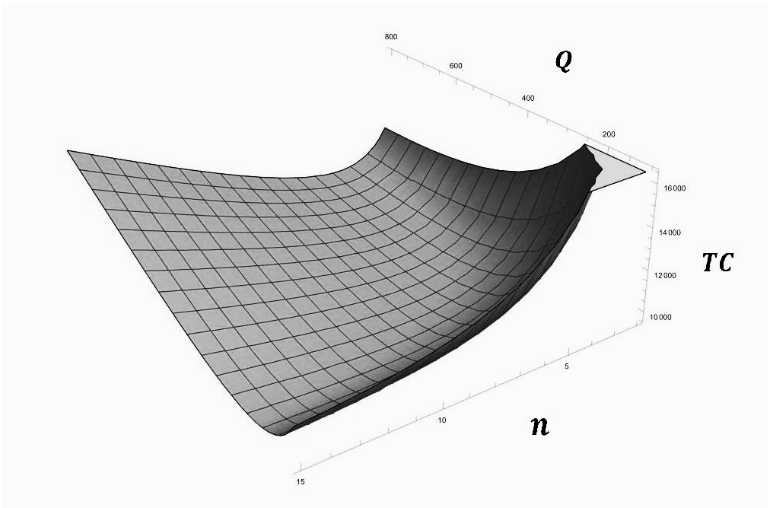

Figure 5: 3D graph of Example 2

Sensitivity Analysis

A sensitivity analysis has been performed to evaluate the effect of variations in inventory parameters on the optimal solution of Numerical Example 1. In this analysis, each parameter is independently varied within a range of -20% to +20%, while all other parameters remain fixed at their initial values. By altering one parameter at a time, this method allows for the investigation of how changes in individual inventory parameters impact the overall system, offering valuable insights into the model's robustness and optimality across different scenarios.

Variation in holding cost, setup cost, and transportation

Table 2: Sensitivity Analysis

+	*% Change*	*TC**	*n**	*Q**
A	-20	107152.49	13	1285
	-10	110663.81	12	1368
	+10	117279.19	11	1526
	+20	120409.24	11	1602
F	-20	113932.46	12	1446
	-10	113969.31	12	1447
	+10	114043.01	12	1449
	+20	114079.83	12	1450

h_w	-20	114274.73	12	1501
	-10	114169.78	12	1506
	+10	113957.59	12	1515
	+20	113846.36	11	1520
a	-20	101997.04	12	1263
	-10	108145.43	12	1439
	+10	119787.93	12	1578
	+20	125343.98	11	1641
b	-20	104211.29	10	1762
	-10	109344.59	11	1619
	+10	118439.29	12	1425
	+20	122565.82	13	1355

cost lead to corresponding adjustments in the optimal cost function, with varying degrees of impact for each parameter. The setup cost demonstrates high sensitivity, indicating that even small fluctuations result in substantial changes to the overall cost. In contrast, holding cost and transportation cost exhibit lower sensitivity, meaning that variations in these parameters have a more moderate effect on the optimal cost function, suggesting a more stable influence on total costs compared to the setup cost. Variations in the demand parameters a and b have a substantial and direct influence on the total cost function. An increase in these parameters leads to a corresponding rise in the total cost, while a decrease results in lower costs. As demand grows, the optimal number of shipments and the lot size per shipment also increase to maintain cost efficiency.

Discussion

This paper presents a comprehensive inventory model aimed at addressing the complexities of modern supply chains. By incorporating factors such as stock-dependent demand, variable holding costs, trade credit policies, and imperfect production, the model offers a realistic framework for managing inventory in a two-warehouse system. The distinction between the vendor's owned warehouse and a rented warehouse is crucial, as it provides a more accurate reflection of real-world inventory management, where holding costs in rented warehouses fluctuate based on the stock volume stored. This dynamic representation of storage costs better aligns with practical scenarios often encountered by companies, making the model more applicable to real-life supply chain operations. The inclusion of a trade credit policy, allowing the vendor to extend a credit period to the buyer, adds another layer of realism by promoting financial flexibility and trust in vendor-buyer relationships. Additionally, accounting for imperfect production processes enables a deeper investigation into how production defects impact system performance and costs. By addressing these real-world factors, the model effectively bridges the gap between theoretical inventory management and the operational challenges businesses face.

CONCLUSIONS

The study's results reveal that setup costs and demand parameters have a more significant impact on total system costs compared to transportation and holding costs. This highlights the importance of optimizing demand management and setup processes to achieve cost-efficient inventory operations. Although transportation and holding costs remain relevant, their lower sensitivity suggests that substantial cost reductions can be realized by focusing on improving demand forecasts and reducing setup-related expenses. This research contributes to the literature by offering an inventory model that better captures the complexities of supply chains subject to variable demand and holding costs. It also provides valuable insights for practitioners, emphasizing the critical role of demand management and setup optimization in minimizing total costs while accounting for real-world factors.

Future Research

For future research, the model could be expanded to include scenarios involving shortages, either fully or partially backlogged, and more complex trade credit policies, such as a two-level credit system. Furthermore, extending the model to multi-buyer systems would enhance its applicability to more complex supply chain environments.

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