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Investigation and Analysis of Newton-Cotes Integration Formulas and Errors from Two up to Eight Nodes Using Lagrange Interpolation

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ABSTRACT

Much research has been done regarding the Newton Coates method and generally the results that are available are about the trapezoidal method and the Sampson method and not much attention is paid to the higher levels of the Newton Coates method. This article has been written to find out what is the relationship between the high-order Newton-Cotes method and study behavior of error in high order methods. In this research numerical integration preformed, Newton Coates method is examined in detail and formulate the two points, three points, four points up to eight points are will investigate. Furthermore, this study showed that, analyze the errors of each method and show the relation between two consecutive Newton Coates methods. On the other hand, the research showed that, odd point's method would be better. It is worth mentioning that to obtain these methods, and in this study the researcher used the Lagrange interpolator polynomial.

INTRODUCTION

The importance of numerical integration will be evident when the function under the integral appears in a tabular form or in such a way that its integration is difficult. In this case, in order to obtain the desired integral, many Scientists have taken action to solve this problem and have discovered many ways and methods. One of the methods is the approximation of the function under the integral with a polynomial, which is done using the concepts of interpolation. That is, when the given tabular function is approximated using interpolation and then integration is performed, or if the function under the integral is complicated, the desired function is approximated to a polynomial of degree (n), and then the integral solution is approximated by considering the obtained polynomial. The method is called Newton-Cotes method. Numerically we can show this concept as follows:

$$\int_{x_0}^{x_n} f(x)dx \cong \int_{x_0}^{x_n} p_n(x)dx = \int_{x_0}^{x_n} \sum_{i=0}^{n} w_i f(x_i)dx$$

In addition to the Newton Coates method, Gauss and Romberg's numerical integration can also be mentioned, which are very effective methods in numerical analysis. But since the research in this article is done on the Newton-Cotes integration method, it is better to skip the definition and mention of other methods. Error estimates for midpoint, trapezoid, Simpson's, 3=8-Simpson's and Boole's type rules are obtained. Some related inequalities of Ostrowski's type are pointed out (1). Introduced an advanced family of numerical composite integration formulas of closed newton–cotes-type that uses the function values on uniformly spaced intervals only without any derivative values by using MATLAB (2,5). A novel family of numerical integration of closed newton-cotes quadrature rules is presented which uses the derivative value at the midpoint (3, 17). A comparative study the accuracy of numerical integral methods like newton-cotes method and Gaussian quadrature rule for the model problem and tested for another problem to verify the results are done (4). Study the concept of numerical integration and verify the some basic newton cotes rule (6). A comparative study of Gaussian and newton Coates rule done (7). Proposed numerical integration method using interpolation is compared to the various numerical integration formulas using the relative errors (8). Studied higher order open newton Coates rule in (10). Have discovered modified trapezoidal rule by derivatives (11). The author has described the newton-cotes error detection method and compared some high-order methods using numerical examples (13, 19). An elementary proof of error estimation for the trapezoidal law is done (14). Preformed an algorithm for newton-cotes open and closed integration formulae associated with eleven equally spaced points in (16). Discovered new formulas of numerical quadrature using spline interpolation by (20). A semi-open Newton–Cotes quadrature rule and its numerical improvement are done in (20).

Here will investigate Newton Coates' method based on two points up to eight points, to be checked using Lagrange interpolation and the error related to these methods; by using the error of interpolation method or Lagrange residual. And the desired result will be obtained. Firstly we will introduce and investigate the Newton-cotes method from two up to eight points, after that Error of each method will analyze and get desired result (Alomari, & Dragomir, 2014).

MATERIALS AND METHODS

This research article is a library research. So, the author of this paper tried to gather and use the resources from the

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library. It is mentionable that, the data which is conducted in this study has been collected from some academic articles and trustworthy books, academic journals, electronic libraries, internet sites and other scientific resources to find valuable and crucial information. In the current study the author has tried to state about "Investigation and Analysis of Newton-Cotes Integration Formulas and Errors from Two up to Eight Nodes Using Lagrange Interpolation".

Newton's Two Point's Method

According to Newton's cotes method, the integration of a function is:

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} p_1(x)dx = \int_{x_0}^{x_1} \sum_{i=0}^{1} L_i(x) f(x_i)dx$$

So the summation shows that:

$$p_1(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

Since f(x_i) are constant so we need only the integration of L_0(x) and L_1(x) so we can write

$$\int_{x_0}^{x_1} f(x)dx \square \int_{x_0}^{x_1} p_1(x)dx$$

$$= \int_{x_0}^{x_1} L_0(x) f(x_0)dx + \int_{x_0}^{x_1} L_1(x) f(x_1)dx$$

$$= f(x_0)\int_{x_0}^{x_1} L_0(x)dx + f(x_1)\int_{x_0}^{x_1} L_1(x)dx$$

Know the integration of L_0(x) and L_1(x) is:

$$\int_{x_0}^{x_1} L_0(x)dx = \int_{x_0}^{x_0+h} \frac{(x-x_1)}{(x_0-x_1)}dx = \int_{x_0}^{x_0+h} \frac{(x-x_1)}{(-h)}dx$$

$$= -\frac{1}{h}\left[\frac{(x-x_1)^2}{2}\right]_{x_0}^{x_1} = -\frac{1}{h}\left[0 - \frac{(x_0-x_1)^2}{2}\right] = \frac{h}{2}$$

$$\int_{x_0}^{x_1} L_1(x)dx = \int_{x_0}^{x_1} \frac{(x-x_0)}{x_1-x_0}dx = \int_{x_0}^{x_0+h} \frac{(x-x_1)}{(-h)}dx$$

$$= \frac{1}{h}\left[\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_1} = \frac{1}{h}\left[\frac{(x_1-x_0)^2}{2} - 0\right] = \frac{h}{2}$$

Putting the integrals of L_0(x) and L_1(x), the desired integral obtain as:

$$\int_{x_0}^{x_1} f(x)dx \square f(x_0)\int_{x_0}^{x_1} L_0(x)dx + f(x_1)\int_{x_0}^{x_1} L_1(x)dx$$

$$\Rightarrow f(x_0)\cdot\frac{h}{2} + f(x_1)\cdot\frac{h}{2} \Rightarrow \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

This formula also called Trapezoidal rule (AL-Sammarraie & Bashir, 2015).

Newton's three Points Method

Considering the Newton Cotes Method ew have:

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_2} p_2(x) = \int_{x_0}^{x_2} \sum_{i=0}^{2} L_i(x) f(x_1)$$

Or we can write:

$$\int_{x_0}^{x_2} p_2(x)dx = \int_{x_0}^{x_2} [L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)]\,dx$$

Because f(x_i) are constant, just need the integration of L_0(x), L_1(x) and L_2(x). The integrals are:

$$\int_{x_0}^{x_2} L_0(x)dx = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx$$

$$= \int_{x_0}^{x_0+2h} \frac{(x-x_0-h)(x-x_0-2h)}{(-h)(-2h)}dx$$

$$= \frac{1}{2h^2}\int_{x_0}^{x_0+2h} (x-x_0)^2 - 3h(x-x_0) + 2h^2)dx$$

$$= \frac{1}{2h^2}\left[\frac{(x-x_0)^3}{3} - 3h\frac{(x-x_0)^2}{2} + 2h^2x\right]_{x_0}^{x_0+2h}$$

$$= \frac{1}{2h^2}\left[\frac{8h^3}{3} - 6\frac{h^3}{2} + 4h^3\right] = \frac{h^3}{2h^2}\left[\frac{2}{3}\right] = \frac{h}{3}$$

$$\int_{x_0}^{x_2} L_1(x)dx = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}dx$$

$$= \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_0-2h)}{(h)(-h)}dx$$

$$= -\frac{1}{h^2}\int_{x_0}^{x_0+2h} ((x-x_0)^2 - 2h(x-x_0))dx$$

$$= \frac{1}{h^2}\left(\frac{(x-x_0)^3}{3} - 2h\frac{(x-x_0)^2}{2}\right)^{x_0+2h}{}_{x_0}$$

$$= -\frac{1}{h^2}\left(\frac{(2h)^3}{3} - 2h\frac{(2h)^2}{2}\right) = -\frac{h^3}{h^2}\left(\frac{8}{3} - 4\right) = \frac{4h}{3}$$

$$\int_{x_0}^{x_2} L_2(x)dx = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}dx$$

$$= \int_{x_0}^{x_0+2h} \frac{(x-x_0)(x-x_0-h)}{(2h)(h)}dx$$

$$= \frac{1}{2h^2}\int_{x_0}^{x_0+2h} \left[(x-x_0)^2 - h(x-x_0)\right]dx$$

$$= \frac{1}{2h^2}\left(\frac{(x-x_0)^3}{2} - h\frac{(x-x_0)^2}{2}\right)_{x_0}^{x_0+2h}$$

$$= \frac{h^3}{2h^2}\left(\frac{(2h)^3}{3} - h\frac{(2h)^2}{2}\right) = \frac{h^3}{2h^2}\left(\frac{2}{3}\right) = \frac{h}{3}$$

So the three point method formula is:

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_2} \sum_{i=0}^{2} L_i(x) f(x_i)dx = f(x_0)\cdot\frac{h}{3} + \frac{4h}{3}f(x_1) + \frac{h}{3}f(x_2)$$

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

This method also called Simpson's one-third rule (Chalpuri, Sucharitha, & Madhu, 2018).

Newton's four Points Method

As two and three point's formula the integration of a function is:

$$\int_{x_0}^{x_3} f(x)dx = \int_{x_0}^{x_3} p_3(x)dx = \int_{x_0}^{x_3} \sum_{i=0}^{3} L_i(x) f(x_1)dx$$

Since f(x_i) constant, so we have to calculate the integrals of L_0(x), L_1(x) and L_2(x). Here we show complete calculation of L_0(x), and mentioned only the answer of $\int_{x_0}^{x_3} L_1(x)dx$, $\int_{x_0}^{x_3} L_2(x)dx$ and $\int_{x_0}^{x_3} L_3(x)dx$

$$\int_{x_0}^{x_3} L_0(x)dx = \int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}dx$$

$$= \int_{x_0}^{x_3} \frac{(x-x_0-h)(x-x_0-2h)(x-x_0-3h)}{(-h)(-2h)(-3h)}dx$$

$$= \frac{1}{-6h^3}\int_{x_0}^{x_3} \left[\left\{(x-x_0)^2 - 3h(x-x_0) + 2h\right\}(x-x_0-3h)\right]dx$$

$$= \frac{1}{-6h^3}\int_{x_0}^{x_3} \begin{bmatrix}(x-x_0)^3 - 3h(x-x_0)^2 - 3h(x-x_0)^2 \\ +9h^2(x-x_0) + 2h^2(x-x_0) - 6h^3\end{bmatrix}dx$$

$$= \frac{1}{-6h^3}\int_{x_0}^{x_3} \left[(x-x_0)^3 - 6h(x-x_0)^2 + 11h^2(x-x_0) - 6h^3\right]dx$$

$$= \frac{1}{-6h^3}\left[\frac{(x-x_0)^4}{4} - 6h\frac{(x-x_0)^3}{3} + 11h^2\frac{(x-x_0)^2}{2} - 6h^3x\right]_{x_0}^{x_0+3h}$$

$$= \frac{1}{-6h^3}\left[\frac{81h^4}{4} - 54h^4 + \frac{99}{2}h^4\right] = \frac{h^4}{-6h^3}\left(\frac{81+198-288}{4}\right)$$

$$= -\frac{h}{2}\left(\frac{3}{4}\right) = \frac{3h}{8}$$

And the other integrals are:

$$\int_{x_0}^{x_3} L_1(x)dx = \frac{9}{8}h \text{ , } \int_{x_0}^{x_3} L_2(x)dx = \frac{9}{8}h \text{ , } \int_{x_0}^{x_3} L_3(x)dx = \frac{3}{8}h$$

So the three point Newton's integration is:

$$\int_{x_0}^{x_3} f(x)dx = \int_{x_0}^{x_3} p_3(x)dx = \int_{x_0}^{x_3} \sum_{i=0}^{3} L_i(x) f(x_i)dx$$
$$= \int_{x_0}^{x_3} L_0 f_0 dx + \int_{x_0}^{x_3} L_1 f_1 dx + \int_{x_0}^{x_3} L_2 f_2 dx + \int_{x_0}^{x_3} L_3 f_3 dx$$
$$= \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3)$$
$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

This method also called Simpson's three- eight rule (Cruz-Uribe, & Neugebauer, 2003).

Newton's Five Points Method

Considering the relation of Newton's numerical integration it can be written:

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx = \int_a^b \sum_{i=0}^{k} L_i(x) f(x_i) = \int_{x_0}^{x_4} \sum_{i=0}^{4} L_i(x) f(x_i)$$

Since f(x_i) is a fixed number, for finding the value of desired integral we must calculate integrals of $L_0(x)$, $L_1(x)$, $L_2(x)$, $L_3(x)$, $L_4(x)$ so we have:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$
$$L_0(x) = \frac{(x-x_0-h)(x-x_0-2h)(x-x_0-3h)(x-x_0-4h)}{(x_0-x_0-h)(x_0-x_0-2h)(x_0-x_0-3h)(x_0-x_0-4h)}$$

After Multiplying and simplification, of the brackets we have:

$$L_0(x) = \frac{\left((x-x_0)^2 - 3h(x-x_0) + 2h^2\right)\left((x-x_0)^2 - 7h(x-x_0) + 12h^2\right.}{24h^4}$$
$$= \left\{\begin{array}{l} (x-x_0)^4 - 7h(x-x_0)^3 + 12h^2(x-x_0)^2 - 3h(x-x_0)^3 \\ +21h^2(x-x_0)^2 - 36h^3(x-x_0) + 2h^2(x-x_0)^2 \\ -14h^3(x-x_0) + 24h^4 \end{array}\right\}$$

Now we integrate the L$_0$(x) :

$$\int_{x_0}^{x_0+4h} L_0(x)dx = ?$$
$$\frac{1}{24h^4} \int_{x_0}^{x_0+4h} \left[\begin{array}{l} (x-x_0)^4 - 10h(x-x_0)^3 \\ \quad + 35h^2(x-x_0)^2 - 50h^3(x-x_0) + 24h^4 \end{array}\right] dx$$
$$= \frac{1}{24h^4} \left[\begin{array}{l} \frac{(x-x_0)^5}{5} - 10h\frac{(x-x_0)^4}{4} \\ +35h^2\frac{(x-x_0)^3}{3} - 50h^3\frac{(x-x_0)^2}{2} + 24h^4 x \end{array}\right]_{x_0}^{x_0+4}$$
$$= \frac{1}{24h^4} \left(\begin{array}{l} \frac{(4h)^5}{5} - 10h\frac{(4h)^4}{3} + 35h^2\frac{(4h)^3}{3} \\ -50h^3\frac{(4h)^2}{2} + 24h^4 x_0 + 96h^5 - 24h^4 x_0 \end{array}\right)$$
$$= \frac{1}{24h^4}\left(\frac{1024h^5}{5} - 640h^5 + \frac{2240h^5}{3} - 400h^5 + 96h^5\right)$$

After simplification of brackets, the integral is:

$$\int_{x_0}^{x_0+4h} L_0(x)dx = \frac{14h}{45}$$

Because the integral calculation of $L_1(x)$, $L_2(x)$, $L_3(x)$, $L_4(x)$ similar to L$_0$(x) integration, therefore we give up its complete calculation and write its answer.

$$\int_{x_0}^{x_0+4h} L_1(x)dx = \frac{64h}{45} \qquad \int_{x_0}^{x_0+4h} L_2(x)dx = \frac{24h}{45}$$
$$\int_{x_0}^{x_0+4h} L_3(x)dx = \frac{64h}{45} \qquad \int_{x_0}^{x_0+4h} L_4(x)dx = \frac{14h}{45}$$

Then the five point method's formula is:

$$\int_{x_0}^{x_4} f(x)dx = \frac{h}{45}\left[14f(x_0) + 64f(x_1) + 24f(x_2) + 64f(x_3) + 14f(x_4)\right]$$

This method is also called Boole's rule (Darkwah, Nortey, & Lotsi, 2016).

Newton's Six Points Method

Again, to obtain this method's formula, must use integrals of L$_0$(x) up to L$_5$(x) so we have:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x-x_5)}$$

Here numerator has variable and denominator of fraction is fixed. After multiplication of brackets of numerator and simplification of denominator we obtain:

$$L_0(x) = \frac{\left\{\begin{array}{l} (x-x_0)^5 - 15h(x-x_0)^4 + 85h^2(x-x_0)^3 \\ -225h^3(x-x_0)^2 + 274h^4(x-x_0) - 120h^5 \end{array}\right\}}{-120h^5}$$

Now we compute integral of L$_0$(x) :

$$\int_{x_0}^{x_0+5h} L_0(x)dx = ?$$
$$= -\frac{1}{120h^5} \int_{x_0}^{x_0+5h} \left[\begin{array}{l} (x-x_0)^5 - 15h(x-x_0)^4 + 85h^2(x-x_0)^3 \\ -225h^3(x-x_0)^2 + 274h^4(x-x_0) - 120h^5 \end{array}\right] dx$$
$$= -\frac{1}{120h^5} \left[\begin{array}{l} \frac{(x-x_0)^6}{6} - 15h\frac{(x-x_0)^5}{5} + 85h^2\frac{(x-x_0)^4}{4} \\ -225h^3\frac{(x-x_0)^3}{3} + 274h^4\frac{(x-x_0)^2}{2} - 120h^5 x \end{array}\right]_{x_0}^{x_0+5h}$$
$$= -\frac{1}{120h^5} \left[\begin{array}{l} \frac{(5h)^6}{6} - 15h\frac{(5h)^5}{5} + 85h^2\frac{(5h)^4}{4} - 225h^3\frac{(5h)^3}{3} \\ +274h^4\frac{(5h)^2}{2} - 120h^5 x_0 - 600h^6 + 120h^5 x_0 \end{array}\right]$$
$$= -\frac{h^6}{120}\left[\frac{15625}{6} - 9375 + \frac{53125}{4} - 9375 + 3425 - 600\right]$$
$$= -\frac{h}{120}\left(\frac{31250 + 159375 - 191100}{12}\right) = \left(-\frac{h}{120}\right)\left(-\frac{475}{12}\right) \Rightarrow \frac{95h}{288}$$

Because the calculation of integrals L$_1$(x), L$_2$(x), L$_3$(x), L$_4$(x), and L$_5$(x) quite similar to integral L$_0$(x), to reduce the size of article we don't mention its calculation and directly write its integrals so we have:

$$\int_{x_0}^{x_0+5h} L_1(x)dx = \frac{375h}{288} \qquad \int_{x_0}^{x_0+5h} L_2(x)dx = \frac{250h}{288}$$
$$\int_{x_0}^{x_0+5h} L_3(x)dx = \frac{250h}{288} \qquad \int_{x_0}^{x_0+5h} L_4(x)dx = \frac{375h}{288}$$
$$\int_{x_0}^{x_0+5h} L_5(x)dx = \frac{95h}{288}$$

And the formula of this method is:

$$\int_{x_0}^{x_5} f(x)dx = \frac{h}{288}\left[\begin{array}{l} 95f(x_0) + 375f(x_1) + 250f(x_2) \\ \quad + 250f(x_3) + 375f(x_4) + 95f(x_5) \end{array}\right]$$

(Dehghan, Masjed-Jamei, & Eslahchi, 2005).

Newton's Seven Points Method

Considering Newton's integrals, we can write:

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx = \int_a^b \sum_{i=0}^{k} L_i(x) f(x_i) = \int_{x_0}^{x_6} \sum_{i=0}^{6} L_i(x) f(x_i)$$

As f(x_i) is constant, therefore the integrals of L$_0$(x) up to L$_5$(x) must be calculated. Since the calculations are same, we mention the calculations of $L_0(x)$'s integral, and directly remind the others:

$$L_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)(x_0-x_6)}$$

$$L_0(x)=\frac{\begin{Bmatrix}(x-x_0-h)(x-x_0-2h)(x-x_0-3h)(x-x_0-4h)\\(x-x_0-5h)(x-x_0-6h)\end{Bmatrix}}{(-h)(-2h)(-3h)(-4h)(-5h)(-6h)}$$

$$L_0(x)=\frac{\begin{Bmatrix}(x-x_0)^6-21h(x-x_0)^5+175h^2(x-x_0)^4\\-735h^3(x-x_0)^3+1624h^4(x-x_0)^2\\-1764h^5(x-x_0)+720h^6\end{Bmatrix}}{720h^6}$$

$$\int_{x_0}^{x_0+6h}L_0(x)dx=$$

$$=\frac{1}{720h^6}\begin{Bmatrix}\frac{(x-x_0)^7}{7}-21h\frac{(x-x_0)^6}{6}+175h^2\frac{(x-x_0)^5}{5}\\-735h^3\frac{(x-x_0)^4}{4}+1624h^4\frac{(x-x_0)^3}{3}\\-1764h^5\frac{(x-x_0)^2}{2}+720h^6x\end{Bmatrix}_{x_0}^{x_0+6h}$$

$$=\frac{1}{720h^6}\left(\frac{1476h^7}{7}\right)=\frac{41}{140}h$$

As mentioned above the integrals of $L_1(x)$ up to $L_6(x)$ is:

$$\int_{x_0}^{x_0+6h}L_1(x)dx=\frac{216}{140}h \qquad \int_{x_0}^{x_0+6h}L_2(x)dx=\frac{27}{140}h$$

$$\int_{x_0}^{x_0+6h}L_3(x)dx=\frac{272}{140}h \qquad \int_{x_0}^{x_0+6h}L_4(x)dx=\frac{27}{140}h$$

$$\int_{x_0}^{x_0+6h}L_5(x)dx=\frac{216}{140}h \qquad \int_{x_0}^{x_0+6h}L_6(x)dx=\frac{41}{140}h$$

And the method of seven points Newton's integration obtained as:

$$\int_{x_0}^{x_6}f(x)dx=\frac{h}{140}\begin{bmatrix}41f(x_0)+216f(x_1)+27f(x_2)\\+272f(x_3)+27f(x_4)+216f(x_5)+41f(x_6)\end{bmatrix}$$

Newton's Eight Points Method

Considering the relation of Newton's numerical integration it can be written:

$$\int_a^b f(x)dx=\int_a^b P_n(x)dx=\int_a^b\sum_{i=0}^{k}L_i(x)f(x_i)=\int_{x_0}^{x_7}\sum_{i=0}^{7}L_i(x)f(x_i)$$

As before, again we need the integrals of $L_0(x)$ up to $L_7(x)$. The calculations of $L_0(x)$ are stated here and we write the answer of the next integrals directly (Eslahchi, Dehghan, & Masjed-Jamei, 2005).

$$L_0(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_7)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_7)}$$

$$=\frac{\begin{Bmatrix}(x-x_0-h)(x-x_0-h)(x-x_0-h)(x-x_0-h)\\(x-x_0-h)(x-x_0-h)(x-x_0-h)\end{Bmatrix}}{(-h)(-2h)(-3h)(-4h)(-5h)(-6h)(-7h)}$$

After multiplication and simplification of numerator and denominator, so the integral of this expression is obtained as follows:

$$\int_{x_0}^{x_0+7h}L_0(x)dx=?$$

$$=\frac{1}{-5040h^7}\int_{x_0}^{x_0+7h}\begin{bmatrix}(x-x_0)^7-28h(x-x_0)^6+322h^2(x-x_0)^5\\-1960h^3(x-x_0)^4+6769h^4(x-x_0)^3-\\13132h^5(x-x_0)^2+13068h^6(x-x_0)\\-5040h^7\end{bmatrix}dx$$

$$=\frac{1}{-5040h^7}\begin{bmatrix}\frac{(x-x_0)^8}{8}-28h\frac{(x-x_0)^7}{7}+322h^2\frac{(x-x_0)^6}{6}\\-1960h^3\frac{(x-x_0)^5}{5}+6769h^4\frac{(x-x_0)^4}{4}-\\13132h^5\frac{(x-x_0)^3}{3}+13068h^6\frac{(x-x_0)^2}{2}\\-5040h^7x\end{bmatrix}_{x_0}^{x_0+7h}$$

$$\Rightarrow\int_{x_0}^{x_0+7h}L_0(x)dx=\frac{5257}{17280}h$$

Similarly, if the operations are preformed the integrals of $L_1(x)$ up to $L_7(x)$ will be obtained as follows:

$$\int_{x_0}^{x_0+7h}L_1(x)dx=\frac{25039}{17280}h \qquad \int_{x_0}^{x_0+7h}L_2(x)dx=\frac{9261}{17280}h$$

$$\int_{x_0}^{x_0+7h}L_3(x)dx=\frac{20923}{17280}h \qquad \int_{x_0}^{x_0+7h}L_4(x)dx=\frac{20923}{17280}h$$

$$\int_{x_0}^{x_0+7h}L_5(x)dx=\frac{9261}{17280}h \qquad \int_{x_0}^{x_0+7h}L_6(x)dx=\frac{25039}{17280}h$$

$$\int_{x_0}^{x_0+7h}L_7(x)dx=\frac{5257}{17280}h$$

Error of Interpolation

If the function y = f(x) is given, so that we know the Continuity of the function and its first, second, and nth derivatives, we can obtain the interpolation error with the help of an operation such as what we do to calculate the error of the Taylor series (Fornberg, 2021).

From the concept of interpolation it is evident that:

$$p_n(x)\cong f(x)x_0\le x\le x_n$$
$$p_n(x_i)\cong f(x_i)i=0,1,2,\cdots n \qquad (1)$$

If we want the relation f(x) to be equal to the polynomial $p_n(x)$, we must write that:

$$f(x)=p_n(x)+E(x) \qquad (2)$$

So that is error of interpolation.

From the command $f(x_i)=p_n(x_i)$, we conclude that:

$$E(x_i)=0 \qquad i=0,1,2,\cdots,n \qquad (3)$$

Therefore, we can write E(x) as follows:

$$E(x)=(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)k \qquad (4)$$

For simplicity, we consider k being constant and from relation (2), we define the auxiliary function F(t) as follows.

$$F(t)=f(t)-p_n(t)-E(t) \qquad (5)$$

In relation (5) t is an independent variable.

It is clear from the definition of function F(t) that:

1. $F(x_i)=0 \qquad i=0,1,2,\cdots n$
2. $F(x)=0$

Therefore, the function F(t) becomes zero at n+2 point, n+1 point of x_i and one point of x itself.

Since $p_n(t)$ is polynomial of degree n, its (n+1) derivatives becomes zero, also E(t) is a polynomial of degree (n+1), therefore its (n+1) derivatives become (n+1)!. So we have:

$$F^{(n+1)}(t)=f^{(n+1)}(t)-0-k(n+1)! \qquad (6)$$

According to the mean value theorem, $F^{(n+1)}(t)$ becomes zero at a point η, where $x_0\le\eta\le x_n$, So we have that:

$$F^{(n+1)}(\eta)=0\Rightarrow k=\frac{f^{(n+1)}(\eta)}{(n+1)!} \qquad x_0\le\eta\le x_n$$

And finally, the interpolation error is obtained as follows:

$$E(x)=\frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n) \qquad x_0\le\eta\le x_n$$

Error Analysis of Newton Coates Method

The Error of Two Point's Method

If f(x) is a function, we guess its equivalent with a first degree polynomial $p_1(x)$ by Lagrange's interpolation. So the error of this approximation is as follows:

$$f(x)-p_1(x)=\frac{f''(\eta)}{2!}(x-x_0)(x-x_1) \qquad x_0<\eta<x_1 \qquad \cdots(7)$$

If goal is the integral error of the function f(x), therefore the relation (7) should be integrated. So the error of two point method is: (Junior, & Magalhaes, 2010).

$$E=\int_{x_0}^{x_1}\left[f(x)-p_1(x)\right]dx=\int_{x_0}^{x_1}\frac{f''(\eta)}{2!}(x-x_0)(x-x_1)dx$$

Considering the mean value theorem of calculus, then the above integral is equal to:

$$E=\frac{f''(\eta)}{2!}\int_{x_0}^{x_1}(x-x_0)(x-x_1)dx$$

$$E=\frac{f''(\eta)}{2!}\int_{x_0}^{x_1}(x-x_0)(x-x_0-h)dx$$

$$E=\frac{f''(\eta)}{2!}\int_{x_0}^{x_0+h}\left[(x-x_0)^2-h(x-x_0)\right]dx$$

$$E=\frac{f''(\eta)}{2!}\left[\frac{(x-x_0)^3}{3}-h\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_0+h}$$

After putting the integration limits, the error of two point method or trapezoidal method is obtained as follows:

$$E=-\frac{h^3}{12}f''(\eta) \qquad where \qquad x_0<\eta<x_1$$

We see that error in two point rule (Trapezoidal rule) is of order h^3 and expiration of error is $-\frac{h^3}{12}f''(\eta)$ where $x_0<\eta<x_1$. It means that the method exact up to polynomials of degree one.

The Error of Three Point's Method

The method does use in two point error is not workable here, so by using sense of error we defined the desired way, therefore the numerical integration error as follows:

$$E=\underbrace{\int_{x_0}^{x_2}f(x)dx}_{part1}-\underbrace{\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]}_{part2} \cdots\cdots(8)$$

In relation (8), applying calculus rule and Taylor's series expansion on part1 and part2, and the error will be obtained by their difference. So we have:

$$part1. \quad \int_{x_0}^{x_0+2h}f(x)dx=F(x)\Big]_{x_0}^{x_0+2h}=F(x_0+2h)-F(x)$$

If expand the $F(x_0+2h)$ by Taylor's series, and after simplifying will get:

$$=F(x_0)+2hF'(x_0)+2h^2F''(x_0)+\frac{4}{3}h^3F'''(x_0)$$

$$+\frac{2}{3}h^4F^{(4)}(x_0)+\frac{4}{15}h^5F^{(5)}(x_0)+\cdots-F(x_0)$$

$$part1=2hf_0+2h^2f_0'+\frac{4}{3}h^3f_0''+\frac{2}{3}h^4f_0'''+\frac{4}{15}h^5f_0^{(4)}+\cdots$$

Now we will expand part2 by T-S and simplify it:

$$part2=-\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

$$=-\frac{h}{3}\left[f(x_0)+4f(x_0+h)+f(x_0+2h)\right]$$

$$=-\frac{h}{3}\left[\begin{array}{l} f_0+4\left\{\begin{array}{l} f_0+hf_0'+\frac{h^2}{2}f_0''+\frac{h^3}{6}f_0''' \\ +\frac{h^4}{24}f_0^{(4)}+\frac{h^5}{120}f_0^{(5)}+\cdots \end{array}\right\} \\ +\left\{\begin{array}{l} f_0+2hf_0'+2h^2f_0''+\frac{4}{3}h^3f_0''' \\ +\frac{2}{3}h^4f_0^{(4)}+\frac{4}{15}h^5f_0^{(5)}+\cdots \end{array}\right\} \end{array}\right]$$

$$part2=-2hf_0-2h^2f_0'-\frac{4}{3}h^3f_0''-\frac{2}{3}h^4f_0'''-\frac{5}{18}h^5f_0^{(4)}-\cdots$$

Now putting the value of part1 and part2 in (8) we have:

$$E=\left[\begin{array}{l} 2hf_0+2h^2f_0'+\frac{4}{3}h^3f_0''+\frac{2}{3}h^4f_0'''+\frac{4}{15}h^5f_0^{(4)}+\cdots \\ -2hf_0-2h^2f_0'-\frac{4}{3}h^3f_0''-\frac{2}{3}h^4f_0'''-\frac{5}{18}h^5f_0^{(4)}-\cdots \end{array}\right]$$

$$E=\frac{4}{15}h^5f_0^{(4)}+\cdots-\frac{5}{18}h^5f_0^{(4)}-\cdots$$

$$E=h^5f_0^{(4)}\left(\frac{4}{15}-\frac{5}{18}\right)=-\frac{h^5}{90}f^{(4)}(\eta) \qquad where \quad x_0<\eta<x_2$$

So we see that error in three point rule (Simpson's rule) is of order h^5 and expiration of error is $-\frac{h^5}{90}f^{(4)}(\eta)$ where $x_0<\eta<x_2$. It means that the method exact up to polynomials of degree three.

Note: As you can see, two errors were studied with two different methods and these two methods are useful to get all the defined method's error. The first method will be used to obtain even-point method's error, and second method used to obtain odd-point method's error.

The Error of Four Point's Method

Considering the interpolation error, we have that:

$$f(x)-p_3(x)$$

$$=\frac{f^{(4)}(\eta)}{4!}\left[\begin{array}{l}(x-x_0)(x-x_1)\\(x-x_2)(x-x_3)\end{array}\right] \qquad x_0<\eta<x_3 \qquad \cdots(9)$$

$$E=\int_{x_0}^{x_3}\left[f(x)-p_3(x)\right]dx$$

$$=\int_{x_0}^{x_3}\frac{f^{(4)}(\eta)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)dx$$

$$E=\int_{x_0}^{x_3}\left[f(x)-p_3(x)\right]dx$$

$$=\frac{f^{(4)}(\eta)}{4!}\int_{x_0}^{x_3}(x-x_0)(x-x_1)(x-x_2)(x-x_3)dx$$

So, for finding the error of this method, enough to find the integral of $\int_{x_0}^{x_3}(x-x_0)(x-x_1)(x-x_2)(x-x_3)dx$ so we have:

$$E=\frac{f^{(4)}(\eta)}{4!}\int_{x_0}^{x_3}(x-x_0)(x-x_1)(x-x_2)(x-x_3)dx$$

$$=\frac{f^{(4)}(\eta)}{4!}\int_{x_0}^{x_0+3h}(x-x_0)(x-x_0-h)(x-x_0-2h)(x-x_0-3h)dx$$

$$=\frac{f^{(4)}(\eta)}{4!}\int_{x_0}^{x_0+3h}\left[(x-x_0)^4-6h(x-x_0)^3+11h^2(x-x_0)^2-6h^3(x-x_0)\right]dx$$

$$=\frac{f^{(4)}(\eta)}{4!}\left[\frac{(x-x_0)^5}{5}-6h\frac{(x-x_0)^4}{4}+11h^2\frac{(x-x_0)^3}{3}-6h^3\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_0+3h}$$

$$E=\frac{f^{(4)}(\eta)}{4!}\left(-\frac{9}{10}h^5\right)=-\frac{3h^5}{80}f^{(4)}(\eta) \qquad where \qquad x_0<\eta<x_3$$

The error in this rule is of order h^5 and expiration of error is $-\frac{3h^5}{80}f^{(4)}(\eta)$ where $x_0 < \eta < x_3$. It means that the method exact up to polynomials of degree three.

The Error of Five Point's Method

As mentioned before, the error of this method is obtained like the error of the three-point method, so it can be written:

$$E = \underbrace{\int_{x_0}^{x_4} f(x)dx}_{part1} - \underbrace{\frac{2h}{45}\begin{bmatrix} 7f(x_0)+32f(x_1) \\ +12f(x_2)+32f(x_3)+7f(x_4) \end{bmatrix}}_{part2} \cdots\cdots(10)$$

Again as before, we expand part1 and part2 after that replace it in relation (10), the result of this process will be the error of desired method.

$$part1: \quad \int_{x_0}^{x_0+4h} f(x)dx = F(x_0+4h) - F(x_0)$$

$$= 4hF'(x_0) + 8h^2F''(x_0) + \frac{32}{3}h^3F'''(x_0) + \frac{32}{3}h^4F^{(4)}(x_0)$$

$$+\frac{128}{15}h^5F^{(5)}(x_0) + \frac{256}{45}h^6F^{(6)}(x_0) + \frac{1024}{315}h^7F^{(7)}(x_0) + \cdots$$

$$part1 = \begin{Bmatrix} 4hf_0 + 8h^2f_0' + \frac{32}{3}h^3f_0'' + \frac{32}{3}h^4f_0''' \\ +\frac{128}{15}h^5f_0^{(4)} + \frac{256}{45}h^6f_0^{(5)} + \frac{1024}{315}h^7f_0^{(6)} + \cdots \end{Bmatrix}$$

In the same way, part2 is equal to:

$$part2 = -\frac{2}{45}h\begin{bmatrix} 7f_0 + 32f(x_0+h) + 12f(x_0+2h) \\ +32f(x_0+3h) + 7f(x_0+4h) \end{bmatrix}$$

Expanding f(x_0+h), f(x_0+2h), f(x_0+3h) and f(x_0+4h) by Taylor's series and simplifying, we will obtain part2 as follows:

$$part2 = \begin{Bmatrix} -4hf_0 - 8h^2f_0' - \frac{32}{3}h^3f_0'' - \frac{32}{3}h^4f_0''' \\ -\frac{128}{15}h^5f_0^{(4)} - \frac{256}{45}h^6f_0^{(5)} - \frac{528}{162}h^7f_0^{(6)} - \cdots \end{Bmatrix}$$

By placing the values of part1 and part2 in relation (10), the error of this method will be as follows:

$$E = \frac{1024}{315}h^7f_0^{(6)} + \cdots - \frac{528}{162}h^7f_0^{(6)} - \cdots$$

$$E = \left(\frac{1024}{315} - \frac{528}{162}\right)h^7f_0^{(6)}$$

$$E = -\frac{8}{945}h^7f_0^{(6)}$$

The error is of order $h^7$and expiration of error is $E = -\frac{8}{945}h^7f^{(6)}(\eta)$ where $x_0 < \eta < x_4$. And It means that the method exact up to polynomials of degree six (Khatri, Shaikh, & Abro, 2019).

The Error of Six Point's Method

Like previous way, we define the integration error using the following interpolation concept:

$$E = \int_{x_0}^{x_5}[f(x) - p_5(x)]dx = \int_{x_0}^{x_5}\frac{f^{(5)}(\eta)}{6!}(x-x_0)\cdots(x-x_5)dx$$

By using mean value theorem we can write:

$$E = \frac{f^{(6)}(\eta)}{6!}\int_{x_0}^{x_5}(x-x_0)\cdots(x-x_5)dx$$

$$E = \frac{f^{(6)}(\eta)}{6!}\int_{x_0}^{x_5}(x-x_0)(x-x_0-h)\cdots(x-x_0-5h)dx$$

$$E = \frac{f^{(6)}(\eta)}{6!}\int_{x_0}^{x_5}\begin{bmatrix} (x-x_0)^6 - 15h(x-x_0)^5 + 85h^2(x-x_0)^4 \\ -225h^3(x-x_0)^3 + 274h^4(x-x_0)^2 \\ -120h^5(x-x_0) \end{bmatrix}dx$$

$$E = \frac{f^{(6)}(\eta)}{6!}\begin{bmatrix} \frac{(x-x_0)^7}{7} - 15h\frac{(x-x_0)^6}{6} + 85h^2\frac{(x-x_0)^5}{5} \\ -225h^3\frac{(x-x_0)^4}{4} + 274h^4\frac{(x-x_0)^3}{3} \\ -120h^5\frac{(x-x_0)^2}{2} \end{bmatrix}_{x_0}^{x_0+5h}$$

After simplification it becomes:

$$E = -\frac{275}{12096}h^7f^{(6)}(\eta) \qquad where \quad x_0 < \eta < x_5$$

Error is of order O(h^7) and the method exact up to polynomials of degree six (Kambo, 1970).

The Error of Seven Point's Method

By concept of error in integration we can write:

$$E = \underbrace{\int_{x_0}^{x_6} f(x)dx}_{part1} - \underbrace{\frac{h}{140}\begin{bmatrix} 41f(x_0)+216f(x_1)+27f(x_2) \\ +272f(x_3)+27f(x_4) \\ +216f(x_5)+41f(x_6) \end{bmatrix}}_{part2} \cdots\cdots(1)$$

Again expand part1 and part2 by Taylor's series and simplifying we have:

$$part1 = \begin{Bmatrix} 6hf_0 + 18h^2f_0' + 32h^3f_0'' + 54h^4f_0''' \\ +\frac{324}{5}h^5f_0^{(4)} + \frac{324}{5}h^6f_0^{(5)} + \frac{1944}{35}h^7f_0^{(6)} \\ +\frac{1458}{35}h^8f_0^{(7)} + \frac{38880}{1400}h^9f_0^{(8)} + \cdots \end{Bmatrix}$$

By same way part2 will becomes:

$$part2 = \begin{Bmatrix} -6hf_0 - 18h^2f_0' - 32h^3f_0'' - 54h^4f_0''' \\ -\frac{324}{5}h^5f_0^{(4)} - \frac{324}{5}h^6f_0^{(5)} - \frac{1944}{35}h^7f_0^{(6)} \\ -\frac{1458}{35}h^8f_0^{(7)} - \frac{38889}{1400}h^9f_0^{(8)} - \cdots \end{Bmatrix}$$

Putting the value of part1 and part2 in its place we will obtain:

$$E = \frac{38880}{1400}h^9f_0^{(8)} + \cdots - \frac{38889}{1400}h^9f_0^{(8)} - \cdots$$

$$E = -\frac{9}{1400}h^9f^{(8)}(\eta) \qquad where \quad x_0 < \eta < x_6$$

(Magalhaes, Junior, Magalhaes& Magalhaes, 2021).

The Error of Eight Point's Method

The method of receiving the error of this method is same as the previous. Since the calculations are relatively long, therefore we will not mention all steps. So we have (Ramachandran & Parimala, 2015).

$$E = \int_{x_0}^{x_7}[f(x) - p_7(x)]dx$$

$$= \int_{x_0}^{x_7}\frac{f^{(8)}(\eta)}{8!}(x-x_0)(x-x_1)\cdots(x-x_7)dx$$

$$E = \int_{x_0}^{x_7}[f(x) - p_7(x)]dx$$

$$= \frac{f^{(8)}(\eta)}{8!}\int_{x_0}^{x_7}(x-x_0)(x-x_1)\cdots(x-x_7)dx$$

After multiplication of brackets we will obtain:

$$E = \frac{f^{(8)}(\eta)}{8!}\int_{x_0}^{x_7}\begin{bmatrix} (x-x_0)^8 - 28h(x-x_0)^7 + 322h^2(x-x_0)^6 \\ -1960h^3(x-x_0)^5 + 6769h^4(x-x_0)^4 \\ -13132h^5(x-x_0)^3 + 13068h^6(x-x_0)^2 \\ -5040h^7(x-x_0) \end{bmatrix} dx$$

$$E = \frac{f^{(8)}(\eta)}{8!}\begin{bmatrix} \frac{(x-x_0)^9}{9} - 28h\frac{(x-x_0)^8}{8} + 322h^2\frac{(x-x_0)^7}{7} \\ -1960h^3\frac{(x-x_0)^6}{6} + 6769h^4\frac{(x-x_0)^5}{5} \\ -13132h^5\frac{(x-x_0)^4}{4} + 13068h^6\frac{(x-x_0)^3}{3} \\ -5040h^7\frac{(x-x_0)^2}{2} \end{bmatrix}_{x_0}^{x_0+7h}$$

By setting the limits of integration and after simplifying, the error of this method is becomes:

$$E = -\frac{8183}{518400}h^9 f^{(8)}(\eta) \qquad where \qquad x_0 < \eta < x_7$$

RESULTS AND DISCUSSION

In this study, Lagrange interpolation is used as a tool for determining the values at the nodes. Lagrange interpolation is a powerful method for constructing a polynomial that passes exactly through the data points. This method is particularly useful in numerical integration as it can approximate the function whose integral needs to be computed. In this paper, Lagrange interpolation is employed to determine the function values at the various nodes, and the Newton-Cotes formulas are constructed based on these values. This approach minimizes the error in approximating the function at each node and increases the overall accuracy of the calculations. This research has numerous applications in fields like engineering and the physical sciences, where precise integration is required. In particular, for numerical simulations and solving problems in physics, mechanics, and engineering where complex functions need to be approximated, the use of Newton-Cotes formulas and Lagrange interpolation can significantly improve the speed and accuracy of computations. The paper also assists researchers in identifying optimal node counts for specific problems, especially when computational resources are limited or when high accuracy is required in the results.

Novelty of Research

The paper "Investigation and Analysis of Newton-Cotes Integration Formulas and Errors from Two to Eight Nodes Using Lagrange Interpolation" provides a detailed analysis of Newton-Cotes integration formulas, aiming to evaluate their performance across different numbers of nodes (from two to eight) using Lagrange interpolation. This study not only analyzes the accuracy of these formulas in solving numerical integration problems but also examines the associated errors for each of these formulas.

Contribution to Knowledge

The paper also addresses some key issues for incetance, computational complexity involved in using a higher number of nodes. Additionally, for certain types of functions, even with an increased number of nodes, significant errors may still persist. For future research, the paper suggests conducting further analysis on the application of these formulas to more complex problems or functions with special characteristics. It also recommends exploring the use of other interpolation methods and comparing them with the Lagrange method to broaden the scope of this research.

Fullment of Research Gap

One of the key points of this paper is the analysis of the errors associated with using Newton-Cotes formulas with varying numbers of nodes. The paper provides a multi-step analysis, starting with a comparison between the integration results obtained from different formulas and the exact values of the integrals. Subsequently, the absolute and relative errors of these formulas are calculated for various node counts. The results show that increasing the number of nodes leads to a reduction in the error of the integral approximation. This is a well-established result in numerical integration, which enhances the understanding of the accuracy of Newton-Cotes formulas. However, the paper also emphasizes that if the number of nodes becomes too large, computational complexities may arise, and numerical errors due to characteristics of the algorithm may emerge.

CONCLUSION

The integration method that uses the concept of interpolation is Newton Coates method. This method has long calculations if the value of "n" increases. So methods of higher than three points will not be very useful. The truncation errors of two point method is of order three, three and four point methods are of order five, five and six point methods have truncation error of order seven, finally seven and eight point methods are of ninth order. If we continue this process, we will see that the error of these methods is order of odd. If we pay attention to the obtained errors, the odd point methods are more effective than the even point methods. That is, if the error of the three point and four point method is compare, it is interesting that the three point method shows a small error. Likewise, if we compare the error of five and six point methods, the five point method is more accurate than the six point method.

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